La Recherche Opérationnelle ou l'art de bien optimiser: un panorama

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What is Operations Research?

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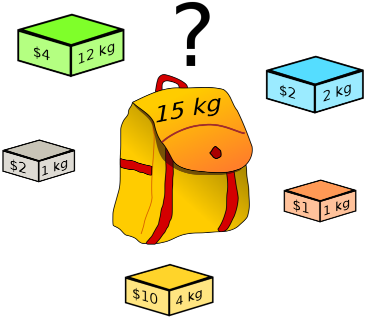
- **Main stream**: make use of mathematics and computer science to build appropriate **models** and **algorithms**.

Starting from an example



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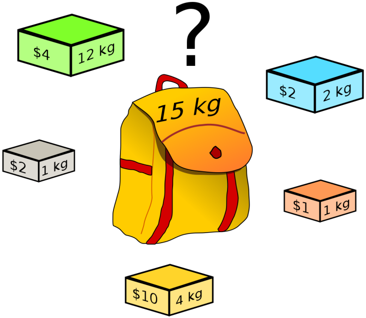

0/1 KNAPSACK

Input: A finite set U, a size $s(u) \in \mathbb{N}$ and value $v(u) \in \mathbb{N}$ for each $u \in U$. A maximum size $B \in \mathbb{N}$.

Goal: Find a subset $U' \subseteq U$ such that

$$\sum_{u \in U'} s(u) \leq B \text{ and } \sum_{u \in U'} v(u) \text{ is maximum.}$$

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- How to solve this kind of problem?
  Exact/optimal algorithms $\Rightarrow$ optimal solutions.
  Heuristic algorithms $\Rightarrow$ "good" solutions.

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- Pt2: What is a good model?

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- Pt1: What is its complexity?
- Pt2: What is a good model?
- Pt3: Which solution algorithm?

What is Complexity Theory?

- Complexity theory provides tools to qualify the time complexity to make a computer solving a problem ([1]),

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- What is the smallest time complexity (in the worst case) a computer can achieve to solve the **0/1 KNAPSACK**?

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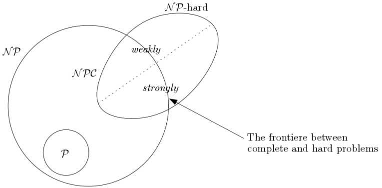
Complexity classes

- The complexity Zoo (`complexityzoo.uwaterloo.ca/Complexity_Zoo`),



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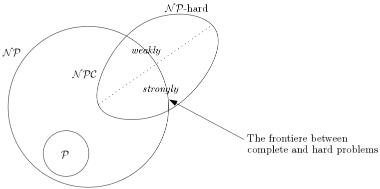

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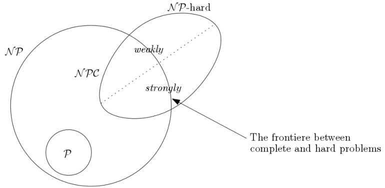

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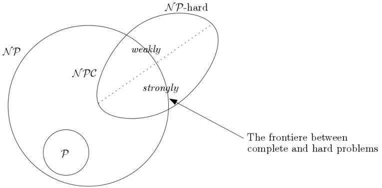

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- $\mathcal{P}$ vs $\mathcal{NP}$? Assumption: $\mathcal{P} \neq \mathcal{NP}$,

  One of the millennium problems of the Clay Mathematics Institute ($1 million reward)

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- For problems shown to be in $\mathcal{NPC}$ ($\mathcal{NP}$-hard problems):
  1. Option 1: Find an optimal algorithm with the "lowest possible" time complexity (though exponential in n)... or at least, fast enough in practice.
  2. Option 2: Find a heuristic algorithm running in polynomial time (so, no guarantee to have the optimal solution).

A scheduling problem

- Getting a model leading to an effective solution is fundamental!

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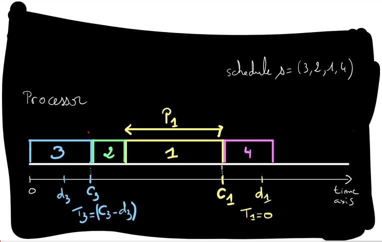
- Consider the following scheduling problem,

Single machine total tardiness (SMTT)

Let be n tasks to perform on a single processor. Each task j is defined by a known processing time p_j and a due date d_j.
For a given schedule s, each task j is given a completion time C_j and a tardiness $T_j = \max(0; C_j - d_j)$.
Find a schedule s with minimum $\sum_j T_j$ value.



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- A position based IP formulation (IP_1),

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\begin{array}{lr}
\text{Minimize } \sum_{k=1}^{n} T_{[k]} & \\
\quad \text{s.t.} & \\
\qquad \sum_{k=1} x_{j,k} = 1 & \forall j = 1, ..., n \\
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\qquad T_{[k]} \geq \sum_{\ell=1}^{k} \sum_{j=1}^{n} x_{j,\ell} p_j - \sum_{j=1}^{n} x_{j,k} d_j & \forall k = 1, ..., n \\
\qquad T_{[k]} \geq 0 & \forall k = 1, ..., n \\
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- For a given instance, provide (IP_1) to a commercial solver (e.g. CPLEX, Gurobi, XPress) and press the *Solve* button!

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- For a given instance, provide (IP_1) to a commercial solver (e.g. CPLEX, Gurobi, XPress) and press the *Solve* button!
- You should be able to solve instances up to about 50 jobs in size.

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- Model the problem differently by making use of Lawler's decomposition [2],

[2] Lawler, E.L (1977). *A pseudopolynomial algorithm for sequencing jobs to minimize total tardiness*, Annals of Discrete Mathematics, 1:331-342.

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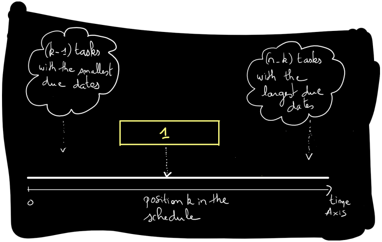
- Illustration on an example,



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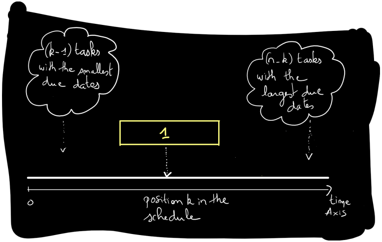

- Dynamic Programming (DP):

$$T[S,0] = \min_{s\leq \ell \leq e}(T[B_\ell,0] + T[A_\ell, \sum_{j\in B_\ell \cup \{j^*\}} p_j] + \max(0; \sum_{j\in B_\ell \cup \{j^*\}} p_j - d_{j^*}))$$

with j^* the longest task in S, B_ℓ (resp. A_ℓ) tasks before j^* (resp. after) when sequenced in position ℓ.

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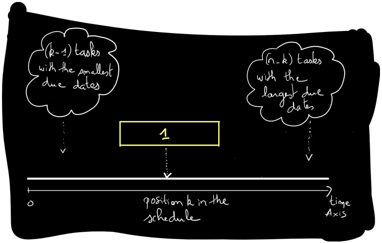

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- You should be able to solve instances up to 100 tasks in size.

Intermediate conclusions

Pit stop

- MP and DP are approaches usable for building models,
- MP and DP can be used also to solve your problem,
- The effectiveness of MP solvers is continuously improved: good challengers for the exact solution of decision/optimization problems.

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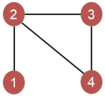

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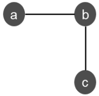

G’

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Let $G = (V, E, \mu, \xi)$ and $G' = (V', E', \mu', \xi')$ be two undirected attributed graphs, with μ (resp. μ') is the set of labels attached to vertices in V (resp. V'), and ξ (resp. ξ') is the set of labels attached to edges in E (resp E').
Let λ be and *edit path*: a minimal set of operations (deletion, insertion, substitution) to transform G into G'.
Find λ^* with minimal cost $d(\lambda) = \sum_{o \in \lambda} c(o)$.

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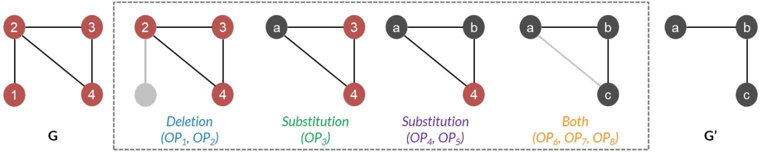

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$$x_{i,k} = \begin{cases} 1 & \text{if } i \in V \text{ is matched with } k \in V' \\ 0 & \text{otherwise} \end{cases}$$

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$|V| \times |V'|$ variables $x_{i,k}$ and $|E| \times |E'|$ variables $y_{ij,k\ell}$.

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$$\sum_{(k,\ell)\in E'} y_{ij,k\ell} \leq x_{i,k} + x_{j,k} \qquad \forall k \in V', \forall (i,j) \in E \quad \text{(C)}$$

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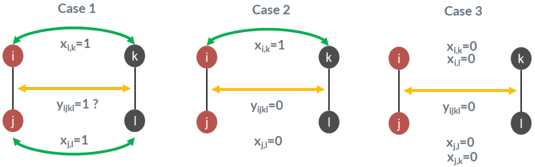
Ex: $y_{ij,k\ell} \leq x_{i,k} + x_{j,k}$ and $y_{ij,k\ell} \leq x_{i,\ell} + x_{j,\ell}$



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  - Objective function:
    Minimize
    $\sum_{i \in V} \sum_{k \in V'} C_v(i,k) x_{i,k} + \sum_{(i,j) \in E} \sum_{(k),\ell \in E'} C_e(ij, k\ell) y_{ij,k\ell} + CSTE$

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- Consider a second IP formulation of the problem (IP2) [4],
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    Same $x_{i,k}$ variables.

$$y_{ij,k\ell} = \begin{cases} 1 & \text{if } (i,j) \in E \text{ is matched with } (k,\ell) \in \tilde{E}' \\ 0 & \text{otherwise} \end{cases}$$

with $\tilde{E}' = E' \cup \{(\ell, k)/(k, \ell) \in E'\}$
$|E| \times |E'|$ variables $x_{i,k}$ and $2 \times |V| \times |V'|$ variables $y_{ij,k\ell}$.

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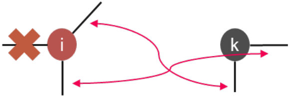

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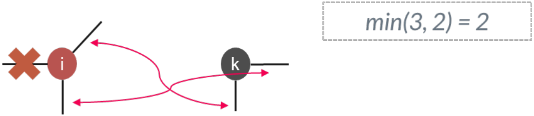

$$\sum_{(i,j)\in E} \sum_{(k,\ell)\in \tilde{E}'} y_{ij,k\ell} \leq d_{i,k} x_{i,k} \qquad \forall i \in V, \forall k \in V' \qquad \text{(D)}$$

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The GED problem

- Consider a second IP formulation of the problem (IP2) [4],
  - Constraints:
    Constraints (A) and (B)
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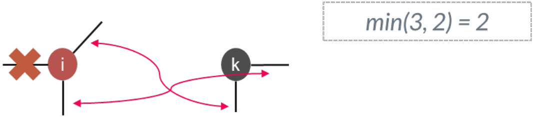

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$|V| + |V'| + |V| \times |V'|$ constraints.

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model (IP1)		model (IP2)	
t_{avg} (s)	$\#Opt$	t_{avg} (s)	$\#Opt$
395.33	25	20.26	25

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880.74	25	604.11	497.07	365	0.70

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Final conclusions

Conclusion

Thinking about the model is as less as important as thinking about solution algorithms.
Designing models occur when dealing with a problem... but also when designing optimization algorithms.

Introduction

- Depending on the complexity of the problem: Exact or Heuristic algorithms,

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- The toolbox of OR (non exhaustive),

Exact algorithms		Heuristic algorithms	
Family	Name	Family	Name
Branching algorithms	Branch-and-Bound	Constructive algorithms	Priority based
	Branch-and-cut		Greedy
	Branch-and-price		Ant CO
	Branch-and-cut-and-price	Branching algorithms	Beam Search
Mathematical Programming	LP		Recovering BS
	MILP		Branch-and-Greed
	QP		Limited Discrepancy Se
	SDP	Neighborhood based algorithms	Simulated Annealing
Dynamic Programming	Forward DP		Tabu
	Backward DP		Multistart
	DP across the subsets		VNS
Constraint Programming			GRASP
Dedicated Approaches			Genetic Algorithms
			Bees Algorithms
		Matheuristics	VPLS
			Local Branching

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2. If your problem is in class $\mathcal{P}$, find the right polynomial-time exact algorithm (dedicated),
3. If your problem is in class $\mathcal{NPC}$, don't search for a polynomial-time exact algorithm: optimality $\Rightarrow$ exponentiality. Do heuristics?

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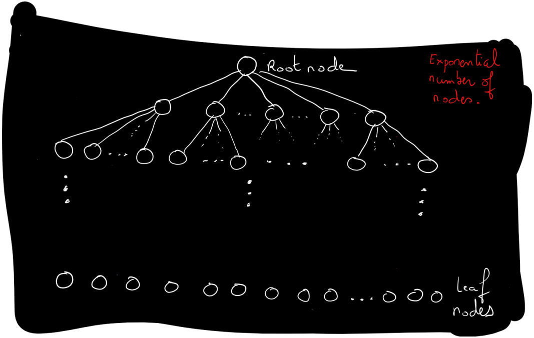
- Try to design a more effective exact algorithm,

  NB: "more effective" means capable of solving to optimality instances of largest size.

Exact solution of the SMTT problem

- Based on Lawler's decomposition, we have designed an exact branching algorithm [5],



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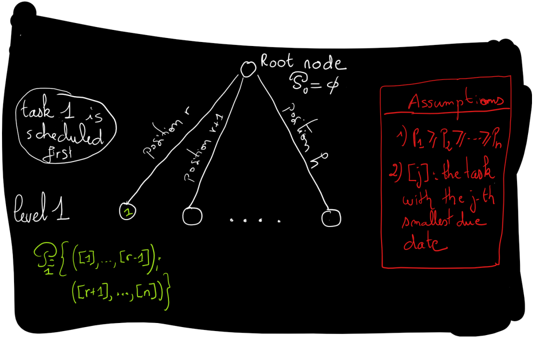

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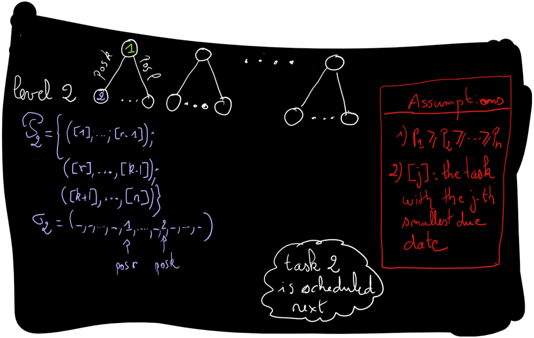

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- In Branch-and-X algorithms, we can also add *cuts*:

If $(C_1(r) > d_{[r+1]})$ then there is no optimal solution in which task 1 is scheduled in position r,

with $C_1(r)$ the completion time of task 1 in the EDD schedule when task 1 is moved to position r.

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- We improve these results by adding a *memorization mechanism*: remember the exploration you have done so far, to avoid exploring useless subproblems in the future.

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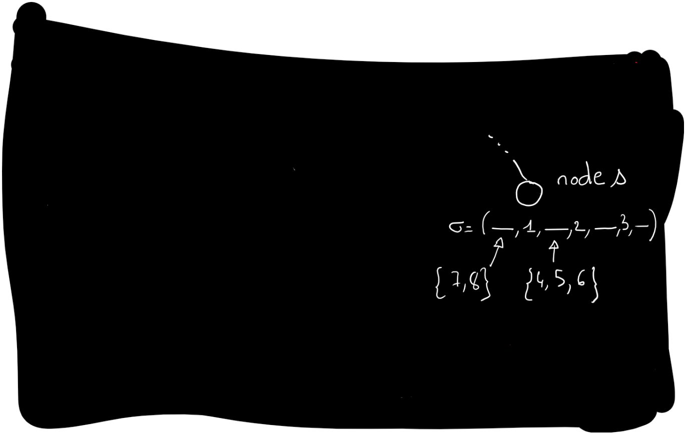

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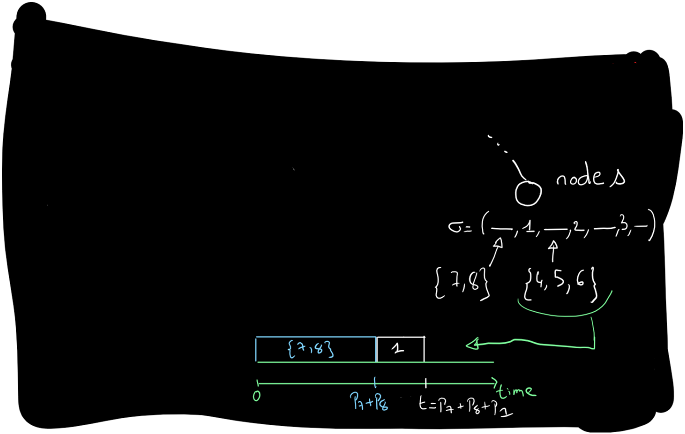

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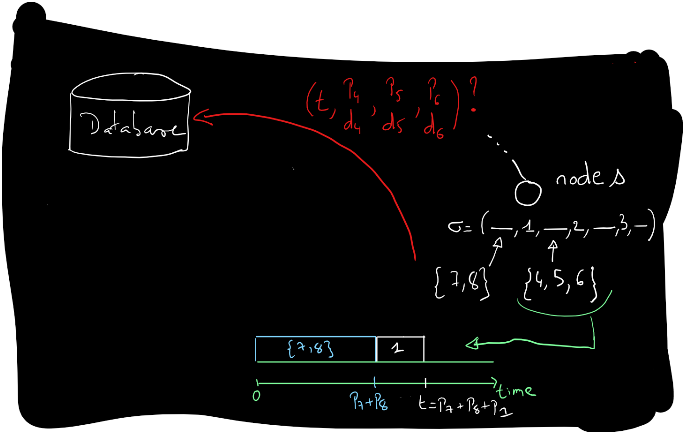

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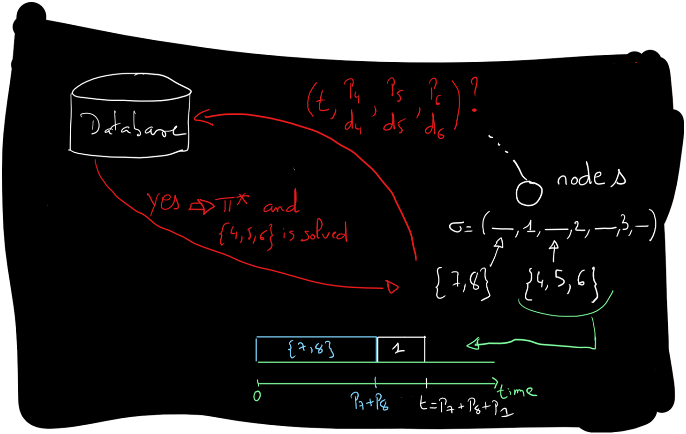

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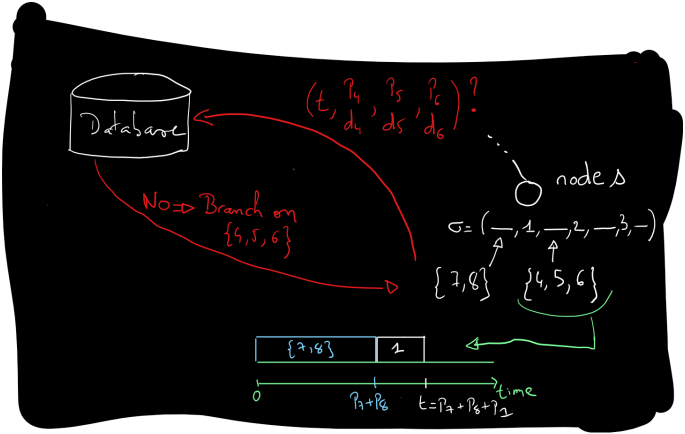

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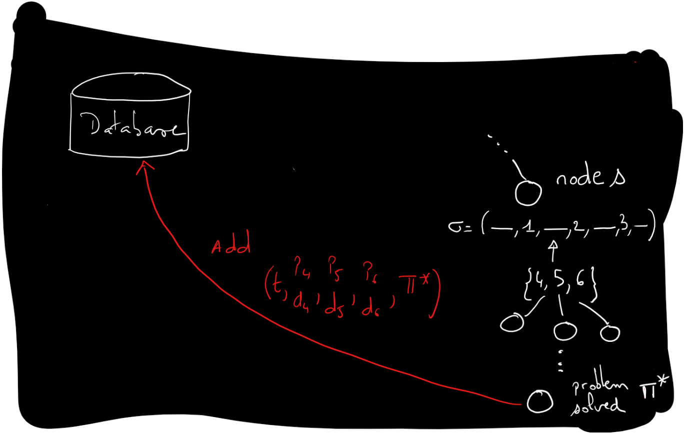

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Pit stop

We have seen so far different exact approaches for $\mathcal{NP}$-hard optimization problems:

- Mathematical Programming (MILP),
- Dynamic Programming (DP),
- Branch-and-X algorithms.

From exact to heuristic approaches

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- Heuristic approaches may become the only option,
- A heuristic algorithm $=$ polynomial running time but no warranty of computing the optimal solution,
- The challenge: (i) find heuristics as close as possible to the optimal solution, (ii) acceptable running time.

Heuristic solution of the GED problem

- Many heuristics have been designed for that problem (see e.g. [4]),

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- Other heuristics exist (some based on branching approaches) but are less efficient than IPFP or GNCCP ([4, 6, 7]),

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- We will see a *neighborhood based heuristic* based on Mathematical Programming: *Matheuristics*.

A Matheuristic for the GED problem

- We design a Local Branching heuristic (LocBra, [8,9])

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$$x_{i,k} = \begin{cases} 1 & \text{if } i \in V \text{ is matched with } k \in V' \\ 0 & \text{otherwise} \end{cases}$$

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- We only work on the $x_{i,k}$ variables,

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  and $S^\ell = \{x_{u,v} \in x^\ell / x_{u,v} = 1\}$.
- Let us denote by $(\text{IP2})^I_\pi(x^\ell)$ the model (IP2) with the constraint $x \in \mathcal{N}(x, x^\ell)$ added,

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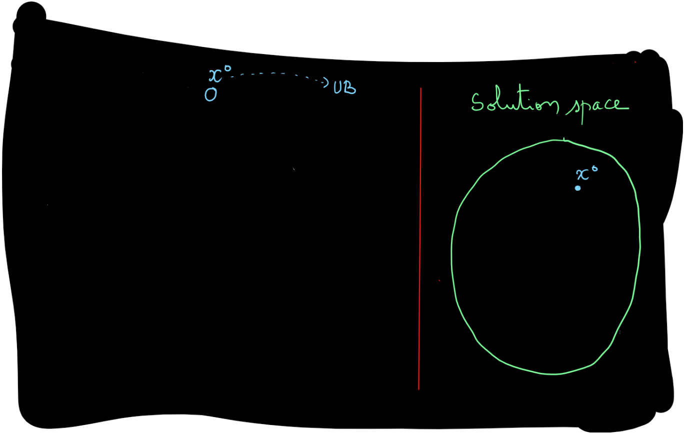

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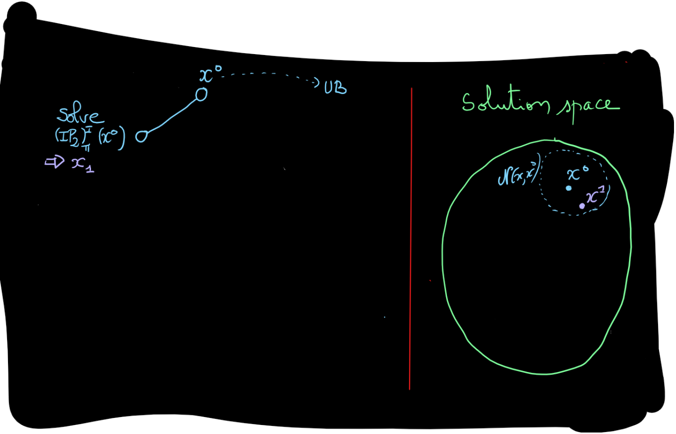

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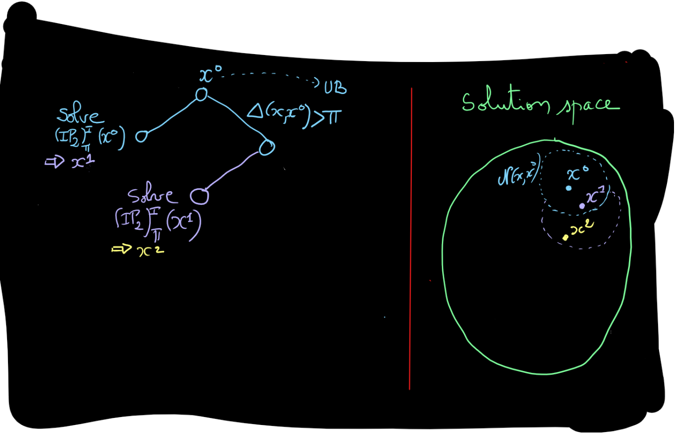

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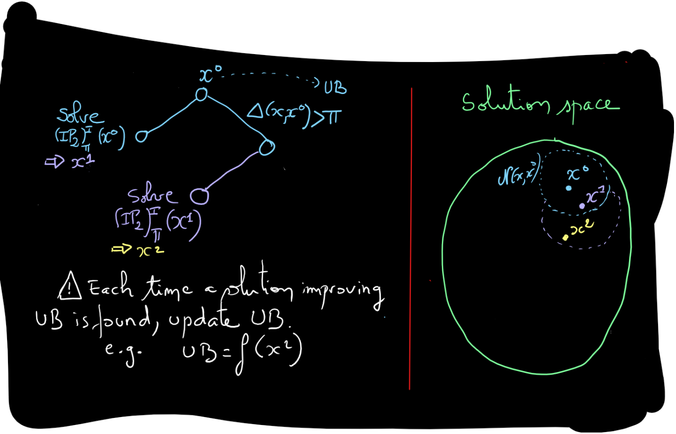

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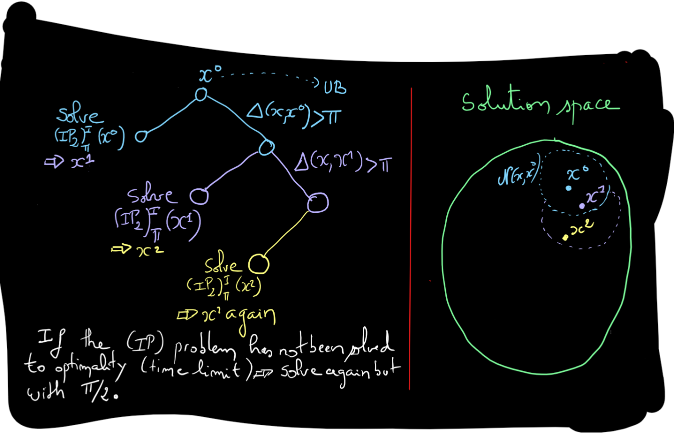

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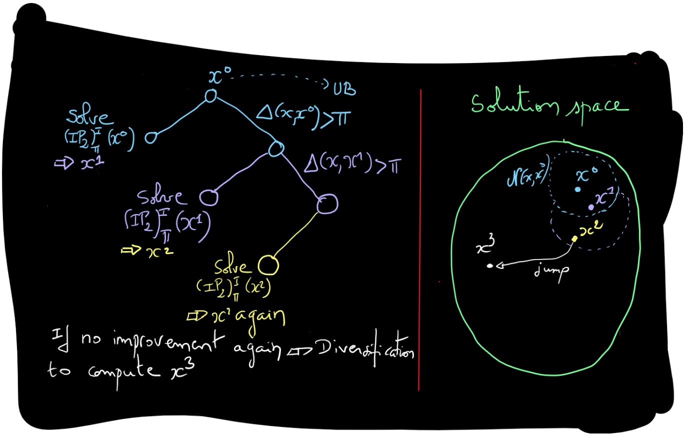

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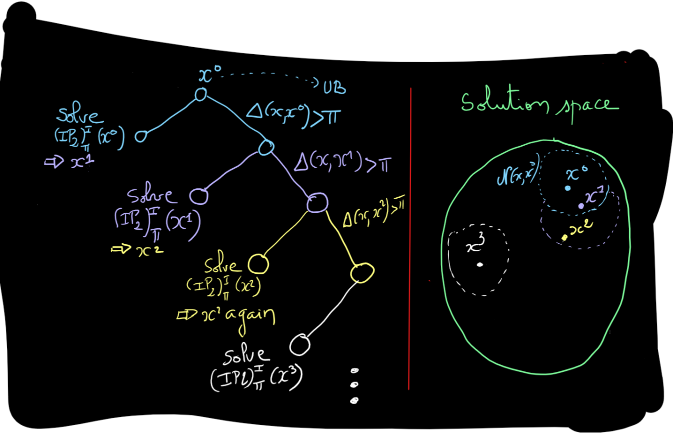

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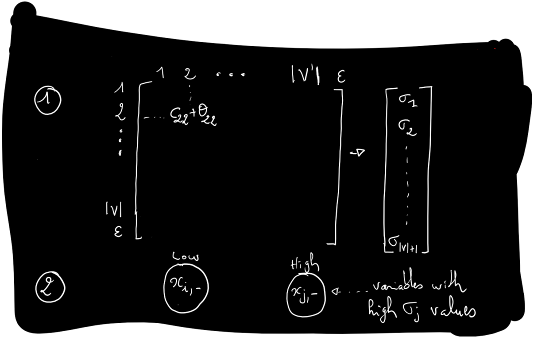
- Compute costs $c_{i,k}$: cost of matching vertices $i \in V$ and $k \in V'$, Compute costs $\theta_{i,k}$: cost of matching edges from i with edges from k (assignment problem),



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- Let S_{div} be the set of variables $x_{i,k}$ with high σ_i values,
- To get a new solution from x^ℓ solve $(\text{IP2})^D_\beta(x^\ell)$:
  - Model $(IP2)$,
  - Add constraint:
    $\Delta'(x, x^\ell) = \left(\sum_{(i,k)\in S^\ell \cap S_{div}} (1 - x_{i,k}) + \sum_{(i,k)\in S_{div} \setminus S^\ell \cap S_{div}} x_{i,k} \right) \geq \beta$

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Size	IPFP		GNCCP		LocBra	
	Avg Dev (%)	Avg time (s)	Avg Dev (%)	Avg time (s)	Avg Dev (%)	Avg time (s)
20x20	1.05	0.09	0.22	2.05	0.06	6.54
30x30	0.98	0.27	0.20	7.21	0.08	8.68
40x40	1.14	0.59	1.68	23.17	0.39	8.82

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  For graphs of size 20x20 and 30x30, we have the optimal solution. For 40x40 we have 63% of optimal solutions.
- A comparison with ground-truth on CMUHOUSE-NA shows that LocBra strongly outperforms the other heuristics (at most 5% of wrong matchings against more than 20% for the others).

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  - Algorithmic issues: choose the right way (exact vs heuristic) and the appropriate algorithm.

Beyond OR

- Some personal thoughts,

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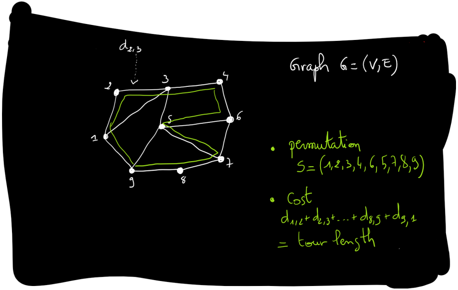
- Let us consider the **Traveling Salesman Problem** (TSP),
  Input: A connected graph $G = (V, E)$ with V the set of vertices (cities) and E the set of edges (routes). Each edge $(i, j) \in E$ is defined by a weight $d_{i,j}$ (distance). We note $n = |V|$.
  Goal: Find a permutation S of vertices such that

$$\left(\sum_{k=1}^{n} d_{S[k],S[k+1]} + d_{S[n],S[1]} \right) \text{ is minimum.}$$



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- The right question should be more **why** using ML,

OR and ML

- Two important reasons ([15]):

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ML and OR should no longer be used separately to solve optimization problems.

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- This would make faster the intensification phase $\Rightarrow$ enable to consider larger neighborhoods $\Rightarrow$ improve the overall LocBra heuristic.

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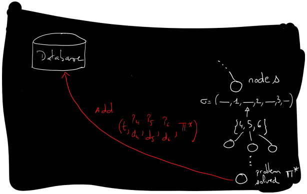

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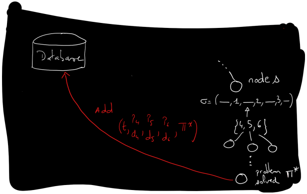

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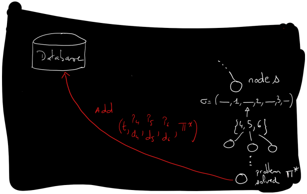

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- Room for ML to learn if a solution will be dominant or not.

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